Partiële Differentiaalvergelijkingen

18 juni 2004

2 $u_x u_y = 1 \qquad u(2x, 0) = 5x \qquad u_x = p,\ u_y = q$

$\hookrightarrow F = pq - 1 = 0 \qquad (x_0(s), y_0(s), u_0(s)) = (2s, 0, 5s)$

$$\begin{cases} x'(\tau) = q \\ y'(\tau) = p \\ u'(\tau) = 2pq = 2 \\ p'(\tau) = 0 \\ q'(\tau) = 0 \end{cases} \qquad \begin{cases} p_0 q_0 - 1 = 0 \\ 2p_0 = 5 \end{cases}$$

$$\Rightarrow p_0 = \frac{5}{2}, \quad \frac{5}{2} q_0 = 1 \Rightarrow q_0 = \frac{2}{5}$$

$$\Rightarrow \begin{cases} p(s,\tau) = \frac{5}{2} \\ q(s,\tau) = \frac{2}{5} \end{cases} \qquad \Rightarrow \begin{cases} x(s,\tau) = \frac{2}{5}\tau + 2s \\ y(s,\tau) = \frac{5}{2}\tau \\ u(s,\tau) = 2\tau + 5s \end{cases}$$

$$\frac{\partial(x,y)}{\partial(s,\tau)} = \begin{vmatrix} 2 & \frac{2}{5} \\ 0 & \frac{5}{2} \end{vmatrix} = 5 \neq 0$$

$$\Rightarrow \frac{5}{2}x = \tau + 5s, \qquad \tau = \frac{2}{5}y$$

$$= \frac{2}{5}y + 5s \qquad 2\tau = \frac{4}{5}y$$

$$5s = \frac{5}{2}x - \frac{2}{5}y$$

$$u = 2\tau + 5s = \frac{4}{5}y + \frac{5}{2}x - \frac{2}{5}y \Rightarrow u(x,y) = \frac{5}{2}x + \cancel{\frac{2}{5}y}$$

controle:

$$u_x u_y = \frac{5}{2} \cdot \frac{2}{5} = 1$$

6 $u_{tt} = u_{xx} + u_{yy} \qquad t > 0$

$u = 0$ op $\partial\Omega$, $\quad u(x,y,0) = f(x)f(y), \quad u_t(x,y,0) = 0$

$$f(s) = \begin{cases} s & 0 < s \leq \frac{\pi}{2} \\ \pi - s & \pi/2 < s \leq \pi \end{cases}$$

neem $\quad u(x,y,t) = X(x)Y(y)T(t)$

$$\Rightarrow X(x)Y(y)T''(t) = X''(x)Y(y)T(t) + X(x)Y''(y)T(t)$$

$$\Rightarrow \begin{cases} \dfrac{T''(t)}{T(t)} = \dfrac{X''(x)}{X(x)} + \dfrac{Y''(y)}{Y(y)} = k_1 + k_2 \\ \dfrac{X''(x)}{X(x)} = \dfrac{T''(t)}{T(t)} - \dfrac{Y''(y)}{Y(y)} = k_1 \\ \dfrac{Y''(y)}{Y(y)} = \dfrac{T''(t)}{T(t)} - \dfrac{X''(y)}{X(y)} = k_2 \end{cases}$$

(6) we hebben dan

$$X''(x) = k_1 X(x)$$
$$Y''(y) = k_2 Y(y)$$
$$T''(t) = (k_1 + k_2) T(t)$$
$$u(x,y,0) = X(x)Y(y)T(0) = f(x)f(y)$$
$$u_t(x,y,0) = X(x)Y(y)T'(0) = 0$$

en $X(0)Y(y)T(t) = X(\pi)Y(y)T(t) = X(x)Y(0)T(t)$

$= X(x)Y(\pi)T(t) = 0 \quad \Rightarrow X(0) = X(\pi) = Y(0) = Y(\pi) = 0$

(we nemen aan dat u niet overal gelijk aan nul is)

$kX = X'' \Rightarrow k\int_0^\pi X^2 dx = \int_0^\pi X''X\,dx = X'X|_0^\pi - \int_0^\pi X'(x)^2 dx = -\int_0^\pi X'(x)^2 dx \Rightarrow k < 0 \Rightarrow$

~~[illegible]~~ we willen ~~[illegible]~~ eindige begrensden nemen

$k_1 = -\lambda_1^2 \quad , \quad k_2 = -\lambda_2^2$ ~~(k_1+k_2) is dan negatief,~~
~~zodat $T(t)$ geen positieve e-macht heeft.~~

$\Rightarrow \quad X(x) = A\cos\lambda_1 x + B\sin\lambda_1 x$

$X(0) = A = 0 \quad \Rightarrow X(x) = B\sin\lambda_1 x$

$X(\pi) = B\sin\lambda_1\pi = 0 \quad \Rightarrow \lambda_1 = n = 1, 2, \ldots$

$X_n(x) = B_n \sin nx$

analoog voor $Y(y) \quad \Rightarrow Y_m(y) = D_m \sin my \qquad m = 1, 2, \ldots$

dan geldt : $T''(t) = -(n^2+m^2)T(t)$

en $T_{nm}(t) = E_{nm}\cos\sqrt{n^2+m^2}\,t + F_{nm}\sin\sqrt{n^2+m^2}\,t$

$T'_{nm}(t) = -\sqrt{n^2+m^2}\,E_{nm}\sin\sqrt{n^2+m^2}\,t + \sqrt{n^2+m^2}\,F_{nm}\cos\sqrt{n^2+m^2}\,t$

$T'_{nm}(0) = \sqrt{n^2+m^2}\,F_{nm} = 0 \Rightarrow F_{nm} = 0 \Rightarrow T_{nm}(t) = E_{nm}\cos\sqrt{n^2+m^2}\,t$

dus $u_{nm}(x,y,t) = X_n(x)Y_m(y)T_{nm}(t)$

$= B_n \sin nx\, D_m \sin my\, E_{nm}\cos\sqrt{n^2+m^2}\,t$

$u(x,y,t) = \sum_{n,m=1}^{\infty} \alpha_{nm}\sin nx \sin my \cos\sqrt{n^2+m^2}\,t$

en $u(x,y,0) = \sum_{n,m=1}^{\infty}\alpha_{nm}\sin nx\sin my = f(x)f(y)$

$\Rightarrow \alpha_{nm} = \left(\frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx\right)\left(\frac{2}{\pi}\int_0^\pi f(y)\sin my\,dy\right)$

$\int_0^\pi f(x)\sin nx\,dx = \int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx$

$$(6) = 2\int_0^{\pi/2} x \sin nx \, dx = 2\left(-\frac{x}{n}\cos nx\Big|_0^{\pi/2} + \int_0^{\pi/2} \frac{1}{n}\cos nx\, dx\right)$$

$$= \begin{cases} -\frac{\pi}{n} & \text{als } n \bmod 4 = 0 \\ \frac{2}{n^2} & \text{als } n \bmod 4 = 1 \\ \frac{\pi}{\cancel{2}n} & \text{als } n \bmod 4 = 2 \\ -\frac{2}{n^2} & \text{als } n \bmod 4 = 3 \end{cases} \quad ?$$

net zo voor $\int_0^{\pi} f(y) \sin my \, dy$, maar dan met m in plaats van n

$$\Rightarrow \alpha_{nm} = \begin{cases} \frac{4}{nm} & \text{als } n \bmod 4 = m \bmod 4 = 0 \text{ of } 2 \\ -\frac{8}{\pi n m^2} & \text{als } n \bmod 4 = 0,\ m \bmod 4 = 1 \text{ of } n \bmod 4 = 2,\ m \bmod 4 = 3 \\ -\frac{4}{nm} & \text{als } n \bmod 4 = 0,\ m \bmod 4 = 2 \text{ of } n \bmod 4 = 2,\ m \bmod 4 = 0 \\ \frac{8}{\pi n m^2} & \text{als } n \bmod 4 = 0,\ m \bmod 4 = 3 \text{ of } n \bmod 4 = 2,\ m \bmod 4 = 1 \\ -\frac{8}{\pi n^2 m} & \text{als } n \bmod 4 = 1,\ m \bmod 4 = 0 \text{ of } n \bmod 4 = 3,\ m \bmod 4 = 2 \\ \left(\frac{4}{\pi n m}\right)^2 & \text{als } n \bmod 4 = m \bmod 4 = 1 \text{ of } 3 \\ \frac{8}{\pi n^2 m} & \text{als } n \bmod 4 = 1,\ m \bmod 4 = 2 \text{ of } n \bmod 4 = 3,\ m \bmod 4 = 0 \\ -\left(\frac{4}{\pi n m}\right)^2 & \text{als } n \bmod 4 = 1,\ m \bmod 4 = 3 \text{ of } n \bmod 4 = 3,\ m \bmod 4 = 1 \end{cases}$$

dus $u(x,y,t)$ met α_{nm} als boven beschreven is de oplossing

$$3u_{yy} + 2u_{xy} + (\sin x)^2 u_{xx} + u_x = 0$$

$$\Rightarrow \quad a = (\sin x)^2$$
$$b = 1$$
$$c = 1 \qquad \Rightarrow \quad b^2 - ac = 1 - (\sin x)^2 = (\cos x)^2 \geq 0 \quad \forall x,y$$

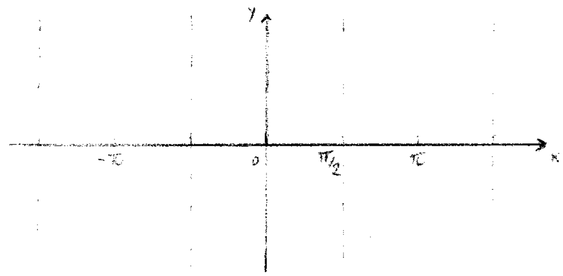
$b^2 - ac > 0$ als $x \neq (k + \frac{1}{2})\pi$

$b^2 - ac = 0$ als $x = (k + \frac{1}{2})\pi \qquad k \in \mathbb{Z}$

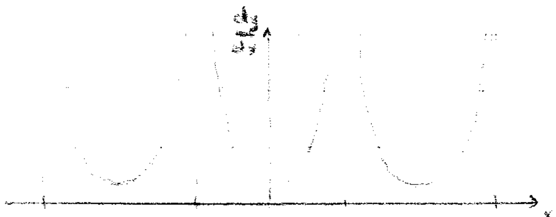
$$\frac{dy}{dx} = \frac{1 \mp \cos x}{(\sin x)^2} \qquad + : \ \frac{1 + \cos x}{(\sin x)^2}$$
$$- : \ \frac{1 - \cos x}{(\sin x)^2}$$

$$b^2 - ac = 0 \Rightarrow \frac{dy}{dx} = \frac{1}{(\sin x)^2}$$

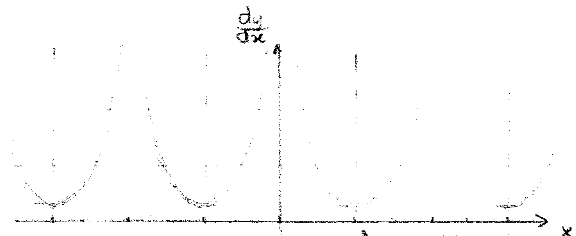
(3)



overal 2 oplossingen, behalve op de stippellijnen, daar één oplossing .



$\frac{dy}{dx} = \frac{1-\cos x}{(\sin x)^2}$



$\frac{dy}{dx} = \frac{1+\cos x}{(\sin x)^2}$

$y(x) = ?$

1 $\quad uu_x + u_y = 1 \qquad u(x,x) = \frac{x}{2}$

$\hookrightarrow (s, s, \frac{s}{2}) = (x_0(s), y_0(s), u_0(s))$

$$\begin{cases} x'(\tau) = u \\ y'(\tau) = 1 \\ u'(\tau) = 1 \end{cases} \Rightarrow \quad \begin{aligned} y(s,\tau) &= \tau + s \\ u(s,\tau) &= \tau + \tfrac{s}{2} \end{aligned}$$

$$\Rightarrow x(s,\tau) = \frac{1}{2}\tau^2 + \frac{s}{2}\tau + s$$

$$\frac{\partial(x,y)}{\partial(s,\tau)} = \begin{vmatrix} \frac{s}{2} & 1 \\ 1 & 1 \end{vmatrix} \neq 0 \quad \text{voor} \quad \frac{s}{2} \neq 1 \iff s \neq 2$$

$$x - y = \frac{1}{2}\tau^2 + \left(\frac{s}{2} - 1\right)\tau$$

$$2(x-y) = \tau^2 + (s-2)\tau = \tau(\tau + s - 2) = \tau(y-2)$$

$$\Rightarrow \tau = \frac{2(x-y)}{y-2}$$

$$y = \tau + s \Rightarrow s = y - \tau = y + \frac{2(y-x)}{y-2} = \frac{y^2 - 2y + 2(y-x)}{y-2} = \frac{y^2-2x}{y-2}$$

$$u = \tau + \frac{s}{2} = \frac{2(x-y)}{y-2} + \frac{1}{2}\,\frac{y^2-2x}{y-2} = \frac{4(x-y)+y^2-2x}{2y-4} = \frac{2x-4y+y^2}{2y-4}$$

$$u(x,y) = \frac{2x-4y+y^2}{2y-4}$$

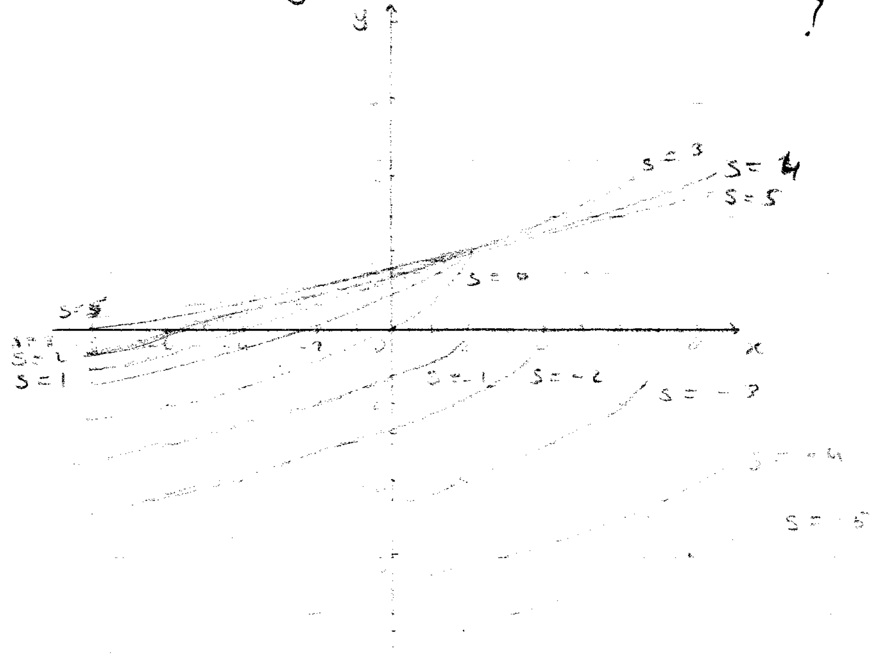
controle : $uu_x + u_y = \cancel{\frac{2x-4y+y^2}{2y-4}\left(\frac{(2y-4)^2 - 2(2x-4y+y^2)}{(2y-4)^2}\right)}$

$$\frac{2x-4y+y^2}{2y-4} \cdot \frac{2}{2y-4} + \frac{(2y-4)^2 - 2(2x-4y+y^2)}{(2y-4)^2} = 1$$

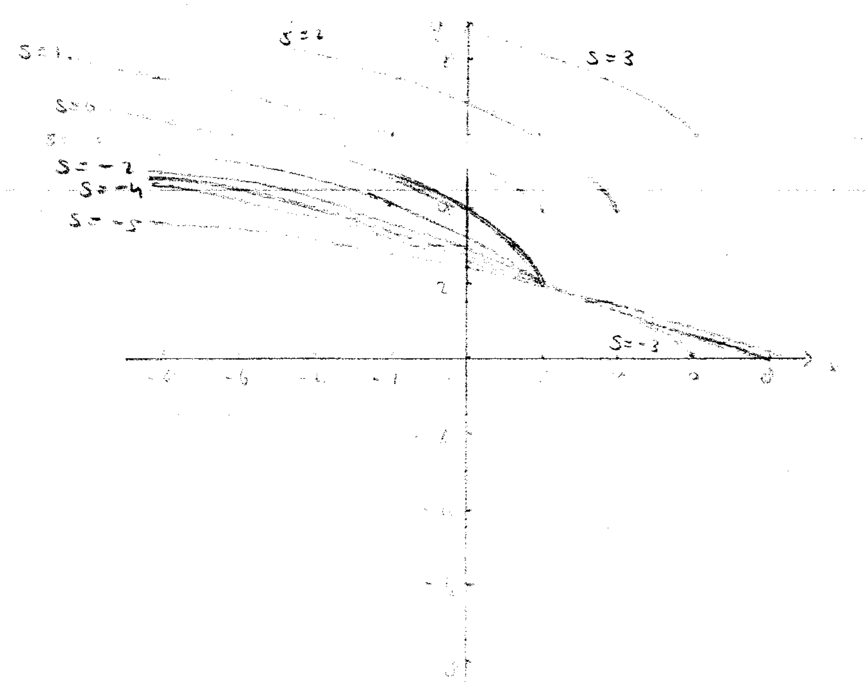
(1) $s = \dfrac{y^2 - 2x}{y - 2} \Rightarrow y_1 = 2 + s - \sqrt{4 + s^2 - 2x}$

$y_2 = 2 + s + \sqrt{4 + s^2 - 2x}$?

y_1



y_2



8 $Lu = u_{xy} + xu_x + yu_y + xyu$

$\Rightarrow a(x,y) = x$

$b(x,y) = y$

$c(x,y) = xy$

$Mv = v_{xy} - (xv)_x - (yv)_y + xyv$

$\begin{cases} v_\eta - \xi v = 0 \\ v_\xi - \eta v = 0 \end{cases} \Rightarrow \begin{cases} v_{\xi\eta} - (\xi v)_\xi - (\eta v)_\eta + \xi\eta v = 0 \\ v(x,\eta) = \cancel{e^{\frac{1}{2}(\eta^2 - y^2)}}\ e^{-\xi(\eta - y)} \\ v(\xi, y) = \cancel{e^{\frac{1}{2}(\xi^2 - x^2)}}\ e^{-\eta(\xi - x)} \end{cases}$

(8) neem $v = e^{xy} u \quad \Rightarrow \quad v_x = yv + e^{xy} u_x \;, \; x v_x = xyv + x e^{xy} u_x$

$$v_y = xv + e^{xy} u_y \;, \; y v_y = xyv + y e^{xy} u_y$$

$$v_{xy} = v + y v_y + x e^{xy} u_x + e^{xy} u_{xy}$$

$$= v + xyv + y e^{xy} u_y + x e^{xy} u_x + e^{xy} u_y$$

$$\Rightarrow v_{xy} - v = e^{xy} u_{xy} + x e^{xy} u_x + y e^{xy} u_y + xy e^{xy} u = e^{xy} L u = 0$$

we hebben dan $\begin{cases} v_{xy} - v = 0 \\ v(x,\eta) = e^{-\xi(\eta - y)} \\ v(\xi, y) = e^{-\eta(\xi - x)} \end{cases}$

4 $u_t = u_{xx} \qquad 0 < x < \pi \qquad t > 0$

$u(0,t) = u(\pi,t) = 0$,

$u(x,0) = f(x)$

neem $u(x,t) = X(x) T(t)$

$\Rightarrow \quad X(x) T'(t) = X''(x) T(t) \quad \Rightarrow \quad \frac{T'}{T} = \frac{X''}{X} = k$

$T' = kT$

$X'' = kX$

$X(0) T(t) = X(\pi) T(t) = 0 \quad \Rightarrow \quad X(0) = X(\pi) = 0$

(we nemen u niet gelijk aan nul)

$$kX = X'' \Rightarrow k \int_0^\pi X^2(x)\,dx = \int_0^\pi X''(x) X(x)\,dx = X'(x) X(x) \Big|_0^\pi - \int_0^\pi X'(x)^2\,dx = -\int_0^\pi X'(x)^2\,dx$$

$\Rightarrow k < 0$, neem $k = -\lambda^2$:

$X = A\cos \lambda x + B \sin \lambda x$

$X(0) = A = 0 \quad \Rightarrow X(x) = B \sin \lambda x$

$X(\pi) = B \sin \lambda \pi = 0 \quad \Rightarrow \lambda = n = 1, 2, \ldots$

$X_n(x) = B_n \sin nx$

$T'(t) = -\lambda^2 T \quad \Rightarrow T(t) = C e^{-\lambda^2 t} \quad \Rightarrow T_n(t) = C_n e^{-n^2 t}$

$u_n = X_n T_n \quad \Rightarrow$

$$u(x,t) = \sum_{n=1}^{\infty} B_n C_n e^{-n^2 t} \sin nx$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n C_n \sin nx = f(x)$$

(4) $\Rightarrow B_n C_n = \frac{2}{\pi}\int_0^\pi f(y) \sin ny \, dy$

$u(x,t) = \sum_{n=1}^\infty \frac{2}{\pi}\int_0^\pi f(y) \sin ny \, dy \; e^{-n^2 t} \sin nx$.

$|f(y)| \leq 1 \Rightarrow \left| \int_0^\pi f(y) \sin ny \, dy \right| \leq \pi$

$\Rightarrow |u(x,t)| \leq 2 \left| \sum_{n=1}^\infty e^{-n^2 t} \sin nx \right| \leq 2 \left| \sum_{n=1}^\infty e^{-n^2 t} \right|$

$\leq 2 \left| \sum_{n=1}^\infty e^{-nt} \right| = 2 \sum_{n=1}^\infty (e^{-t})^n \leq 2(1-e^{-t})^{-1}$

$$f(x) = \begin{cases} 0 & x \in (0, \frac{\pi}{2}] \\ 1 & x \notin (0, \frac{\pi}{2}] \end{cases}$$

$$\int_0^\pi f(y) \sin ny \, dy = \int_{\pi/2}^\pi \sin ny \, dy = \left[-\frac{1}{n} \cos ny \right]_{\pi/2}^{\pi}$$

$$= \begin{cases} \frac{1}{n} & \text{als } n \text{ oneven} \\ 0 & \text{als } n \bmod 4 = 0 \\ -\frac{2}{n} & \text{als } n \bmod 4 = 2 \end{cases}$$

brrr

$$\text{dus } u(x,t) = \begin{cases} \frac{2}{\pi}\sum_{n=1}^\infty \frac{1}{n} e^{-n^2 t} \sin nx & \text{als } n \text{ oneven} \\ 0 & \text{als } n \bmod 4 = 0 \\ -\frac{2}{\pi}\sum_{n=1}^\infty \frac{2}{n} e^{-n^2 t} \sin nx & \text{als } n \bmod 4 = 2 \end{cases}$$

7 $(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0$

neem $u(x,y) = X(x) + Y(y)$

en $X(0) = X'(0) = Y(0) = Y'(0) = 0$

$u_x = X'(x) \qquad u_y = Y'(y) \qquad u_{xy} = 0$

$u_{xx} = X''(x) \qquad u_{yy} = Y''(y)$

$\Rightarrow \left(1 + Y'(y)^2\right) X''(x) + \left(1 + X'(x)^2\right) Y''(y) = 0$

$\Rightarrow \dfrac{X''(x)}{1+X'(x)^2} = -\dfrac{Y''(y)}{1+Y'(y)^2} = k$

dan hebben we $\begin{cases} X''(x) = k\left(1+X'(x)^2\right) \\ Y''(x) = -k\left(1+Y'(x)^2\right) \end{cases}$

(7) hier volgt dat $X(x) = -\frac{\log(\cos(kx+c_1))}{k} + c_2$

$X'(x) = \tan(kx+c_1)$

$X'(0) = \tan c_1 = 0 \Rightarrow c_1 = n\pi \quad , \quad n = 1, 2, \ldots$

$X_n = -\frac{\log(\cos(kx+n\pi))}{k} + c_2$

$X_n(0) = -\frac{\log(\cos n\pi)}{k} + c_2 \Rightarrow \cos n\pi$ moet 1 zijn

n even

$X_n(0) = c_2 = 0 \Rightarrow X_n(x) = -\frac{\log(\cos(kx+n\pi))}{k}$

verder geldt $Y(y) = \frac{\log(\cos(ky-c_3))}{k} + c_4$

$Y'(y) = -\tan(ky-c_3)$

$Y'(0) = \tan c_3 = 0 \Rightarrow c_3 = n\pi \quad , \quad n = 1, 2, \ldots$

$Y_n(x) = \frac{\log(\cos(ky-n\pi))}{k} + c_4$

$Y_n(0) = \frac{\log(\cos(-n\pi))}{k} + c_4 \Rightarrow \cos(-n\pi) = \cos n\pi$ moet 1 zijn

n even

$\hookrightarrow Y_n(0) = c_4 = 0$

$\Rightarrow Y_n(x) = \frac{\log(\cos(ky-n\pi))}{k}$

$u_{nm}(x,t) = X_n(x) + Y_m(y) \quad n, m$ even

$= -\frac{\log(\cos(kx+n\pi))}{k} + \frac{\log(\cos(ky-m\pi))}{k}$

$= \frac{\log\left(\frac{\cos(ky-n\pi)}{\cos(kx+m\pi)}\right)}{k}$

$u(x,t) = \sum_{\substack{n,m=1 \\ n,m \text{ even}}}^{\infty} k^{-1} \log\left(\frac{\cos(ky-n\pi)}{\cos(kx-m\pi)}\right)$

5 $\Delta u - u^3 = 0$ in Ω

en $u = f$ op $\partial\Omega$

stel u_1 en u_2 zijn oplossingen

neem $w = u_1 - u_2$, dan geldt $w = 0$ op $\partial\Omega$

en $\Delta w - (u_1^3 - u_2^3) = 0$

$0 = \int_\Omega w\Delta w - w(u_1^3 - u_2^3)\, dx = \int_{\partial\Omega} w \frac{\partial w}{\partial \nu}\, d\sigma - \int_\Omega (|\nabla w|^2 + w(u_1^3 - u_2^3))\, dx$

(5) $$= -\int_\Omega \left(|\nabla w|^2 + w(u_1^3 - u_2^3)\right) dx$$

$$\int_\Omega \left(|\nabla w|^2 + (u_1-u_2)(u_1^3-u_2^3)\right) dx = 0$$

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$$\Rightarrow |\nabla w|^2 + (u_1-u_2)(u_1^3-u_2^3) = 0$$

$$(u_1-u_2)(u_1^3-u_2^3) = -|\nabla w|^2 \leq 0$$

maar $(u_1-u_2)(u_1^3-u_2^3) \geq 0$ voor $u_1, u_2 \in \mathbb{R}$

dus $(u_1-u_2)(u_1^3-u_2^3) = 0 \Rightarrow u_1 = u_2$

$\Rightarrow$ het probleem kan niet meer dan één oplossing hebben